What Do We Mean by Logical Consequence?

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Introduction

Beginning in the mid-19th century, many prominent philosophers and mathematicians like Frege, Russell, Hilbert felt that mathematics needed a rigorous foundation in logic. The standard approach of the time is called the syntactic or proof-theoretic approach to logic. This says that "for one sentence to be a logical consequence of [a set of premises] is simply for that sentence to be derivable from [them] by means of some standard system of deduction"(2). However, Many results of the time, including Gödel's incompleteness theorems led logicians like Tarski to define logical consequence in a semantic, or model-theoretic way. This eventually became the standard approach for much of the 20-th century. Many have written on the effectiveness of this definition, but in 1990, John Etchemendy offered a fundamental criticism of the "Tarskian Orthodoxy"(5). My research focused on Etchemendy's book and various responses from prominent philosophers.

History

Mathematics and logic have a long and shared history. One aspect of both of these subjects that has long been recognized is that a mathematical or logical claim should be justified by intuitive concepts. For example, in Euclid's Elements, one of his assumption is that "All right angles are equal". This statement seems so obvious we take it for granted. Now, based on only 5 assumptions, *all* of standard geometry can be deduced. This is quite remarkable! This idea that we start from a few intuitively obvious assumptions, called 'axioms' and some simple rules that allow us to derive a conclusion is known as *proof theory*. A proof is simply a set of instructions which tells us how to get from the premises to the conclusion based on these simple rules, known as *rules of inference*.

This seems a bit abstract, but we actually use these rules all the time, especially in mathematics without even thinking about it. For example, if "Roses are red" and "Violets are blue", then "Roses are red and violets are blue". This gives us the intuitive rule, "given two sentences '*p*' and '*q*', we can derive the sentence '*p* & *q*'." One important feature of this "& - rule" is that it is "truth-preserving". This means that, if our premises are true, then our conclusion is also true. All of our rules of inference should, at the very least have this property.

The examples given are quite simple, but proof-theoretic systems can get much more complex. Just think about your calculus class! But even the most complex field of mathematics follows this template. If the axioms are intuitively obviously true, and the rules of inference are intuitively obviously truth-preserving, then anything we can derive from the axioms through the rules of inference must also be true.

There was, however, another Greek with a different approach to logic. In his *Prior Analytics*, Aristotle introduced his theory of the syllogism. This system was quite different from Euclid's system, and instead of using rules to decide which arguments were correct, we look for instances of arguments which violate the principle of truth preservation. For example, the argument "Some people wear socks, and some people wear shoes. Therefore some people wear socks and shoes" seems reasonable, but is not a logical argument, since the argument "Some people are tall, some people are short. Therefore some people are both tall and short." has the same *logical form*, but, while the premises are true, the conclusion is false.

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Tarski, Gödel, and Etchemendy

By the early 20-th century, philosophers had developed a complex but powerful class of logical systems called *first-order theories*. For these systems, there were two established methods to determine the logical validity of an argument. The first approach, following Euclid, became known as proof theory, while the second approach, following Aristotle, became known as model theory. It was well known that, any argument in any first order system which was proof-theoretically valid was also model-theoretically valid. But showing that a model-theoretically valid argument could be proven was a much harder problem. In 1929 Kurt Gödel published his Completeness Theorem, which finally established that for first-order systems, proof theory and model theory produced exactly the same results. This gave the logicians of the time further evidence that we were close to fully solving the problem of logical consequence. But it was only a two years later that Gödel established the now famous Incompleteness theorems. Essentially, the first theorems says that, if our logical system is slightly more complex than a first-order system, capable of just basic arithmetic, then the system will always have statements which cannot be proved or disproved, while the second theorem says that there is no way to fix this problem. These incompleteness theorems are undoubtedly some of the most important results in philosophy of logic.

Since then, modern logic have become an extremely wide and varied field (see 4, ch.1). One of the most influential people in shaping modern logic was Alfred Tarski. In a short article called *On the Concept of Logical Consequence*, Tarski outlined what would eventually be developed into the standard model theory. A majority of the proof-theoretic and model-theoretic approaches that had been developed before are called syntactic systems. This meant that the logical systems divide the terms of a language into categories, treating them as variables. Instead, Tarski proposed the use of a semantic system. This is a system that takes the meaning of the terms in a language into account. This eventually became the standard system of logical consequence for much of the latter half of the 20-th century.

There were many responses to Tarski's account, but in 1990, John Etchemendy wrote *The Concept of Logical Consequence*, which offered a fundamental criticism of the Tarskian account of logical consequence. Model-theoretic semantics branched into roughly two schools of thought. The first is what Etchemendy calls representational semantics. This says that an argument is logically valid if it is impossible for the premises of the argument to be true, while the conclusion is false. It is quite difficult, however, to pin down exactly what should be counted as a possible situation. For example, is the argument "Roses are (completely) blue, therefore roses are not (completely) red" logically valid? If roses really were blue, it is arguably *metaphysically impossible* for roses to simultaneously be red. However, Etchemendy argues that Tarski's real goal was what he calls interpretational semantics.

To illustrate interpretational semantics, he first outlines a similar but slightly simpler system called substitutional semantics first developed by Bolzano. In this system, we establish the logical validity of an argument in a similar way to Aristotle. For example, the Aristotelean syllogism

All books are blue objects
All blue objects are cold
Therefore all books are cold

is valid. We can establish this in a substitutional semantics by first choosing a set of fixed terms. Traditionally, these are logical connectives, like "and", "or", "not", "if... then...", "all", and "some". However, as many have pointed out (Sher, Hanson, Priest), Tarski was wary of choosing these specific terms as being always fixed, as there doesn't seem to be any particular features these terms have which other terms do not. Gila Sher, on the other hand argues that there is a criteria for selecting logical terms, but this is not embraced by everyone. In the argument above, the premises (the first two sentences) are false, but if we replace, or substitute the terms 'books', 'blue objects', and 'cold', with 'philosophers', 'human', and 'mortal' respectively, we get a sentence in which the premises and conclusion are true. In fact, we can reinterpret these term in any way and the argument remains truth preserving. For some technical reasons, we don't use this substitutional semantics, but this is the basic idea behind standard model theory.

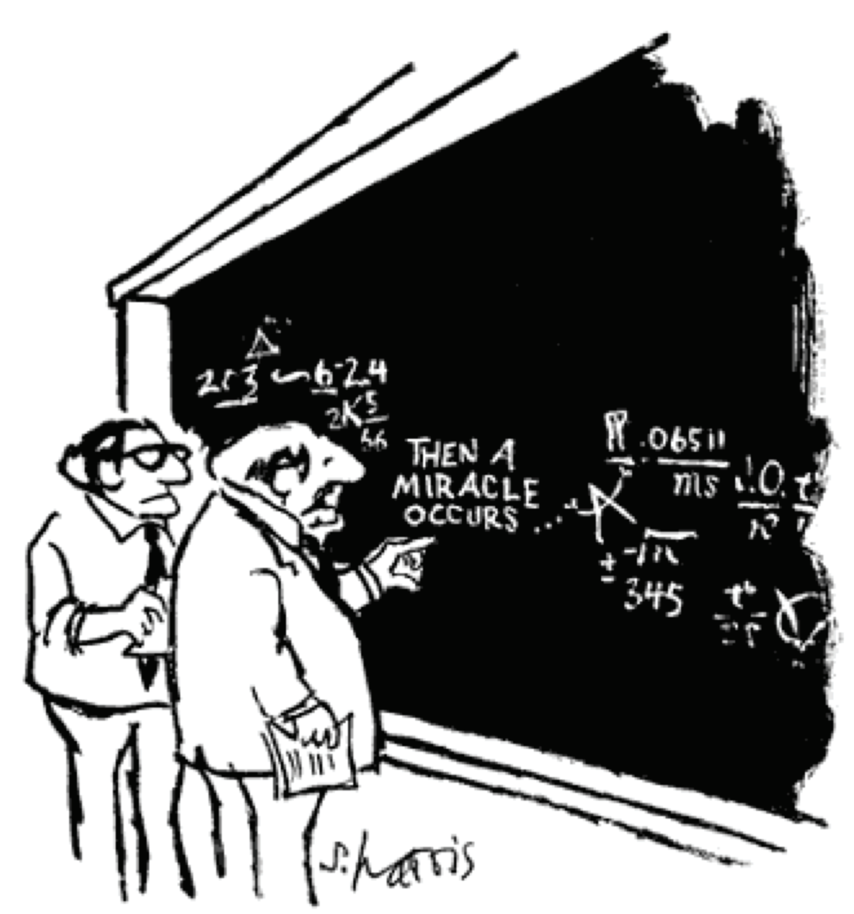
Those who are familiar with formal logic may notice that this is not quite the standard definition of logical consequence. The standard definition says that: A sentence is a logical consequence of a set of premises if, under every interpretation, either some of the premises are false, or the conclusion is true. But in the standard definition of logical consequence in mathematical logic, we use the slightly more complex set theory. Then an interpretation of predicates, for example, can be defined in terms of associating subsets of the "universe of discourse", relations are interpreted as sets of ordered pairs, and so on. (For more details see *Logic for Philosophers* (1) by our very own Professor Cannon!). Although Sher defends the use of set theory in defining model theory, depending on your philosophical stance, this is highly suspicious. After all, how can we use a mathematical theory to define a logical theory? Shouldn't logic precede mathematics?

Etchemendy's main argument revolves around Tarski's use of the word "true". Tarski actually provided a general outline for an "adequate definition of truth" he called Convention T. Although it is somewhat contested, Etchemendy argues that the standard interpretational semantics is a material one. Tarski himself requires that a definition of logical consequence be "materially adequate". This means that, when we say the conclusion is true, we mean the conclusion is true *as a matter of fact*. This is generally what we mean when we say true, but if we contrast this with representational semantics, you may see that this is not something we can take for granted. But this leads to what Etchemendy calls Tarski's Fallacy. If an argument is truth preserving under all interpretations (substitutions of the variable terms, e.g. "books" in the previous argument), then we are actually only looking at all *materially* truth-preserving interpretations. But this can lead to some strange results. In the standard symbolization, formulas like

$$\exists x \exists y (x \neq y)$$
$$\exists x \exists y \exists z ((x \neq y) \& (x \neq z) \& (y \neq z))$$

are made up of only logical terms and variables, but make claims about the *size of the universe*. The first says that there exists at least two distinct objects, while the second says there exists at least three. The fact that they are made up of only traditionally logical, or fixed terms means that they are true under every interpretation given that there *really is* more than 1 object in the universe or there *really are* more than two objects. But this means that both of these statements are logically true! It is clearly true that there are, in fact more than two objects, but is it *logically* true that there are more than two objects?

One of Etchemendy's main criticism of Tarski's model theory is that we can construct arguments like these which are materially truth preserving under all interpretations, i.e. are logically valid according to Tarski's interpretational semantics, but clearly depend on extralogical facts, like the number of objects in the universe. The standard model theory avoids this by using set theory, but using set theory to define logic requires some philosophical maneuvers which are difficult to reconcile with our intuitive understanding of logic and mathematics. So what do we do? Can this problem ever be fixed? This question is still a topic of fierce debate amongst philosophers today.



"I think you should be more explicit here in step two."

A simple proof that we fully understand logical consequence

What is the role of intuition?

One of the aspects of logic that interests me is understanding the connection between our intuitive idea of logic, and the formal understanding of logic. Most, if not all books on modern logic stress the importance of intuition. There is no doubt that simple logical inferences are intuitively obvious. This intuitive idea of 'obviousness' has always guided the study of logic, but this is somewhat circular. If logical consequence is a relation which holds between premises and a conclusion "absolutely", then there may be arguments which are "really" valid, and yet we cannot construct a system which is capable of establishing this fact. So what role does intuition play? Can we only understand logic throught our intuition?

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